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On the Mean Field Theory Bound on the Magnetization

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The mean field bound on magnetization is proved for a class of one-component ferromagnetic systems and for D components systems with arbitrary D.

KEY WORDS: Ferromagnetic models, magnetization, mean field bound, Jensen's inequality, convexity, Bessel functions.

1. INTRODUCTION

The mean field theory bounds on the critical temperature and on the magnetization is a subject of a number of recent works. The critical temperature bounds in the spin-1/2 case were first proved by Griffiths. More recent results are discussed by Simon.[7]

Here we prove magnetization bounds for systems with ferromagnetic two-body interactions. In case of a one component the bounds apply to the natural class $\mathfrak{M}$ of systems,[1,4] (cf. Section 2), for which the critical temperature bounds have been proved in Ref. 1. Using inequalities relating many component systems to one-component ones,[4] we obtain the magnetization bounds for any number of components. Pearce,[4] proves the magnetization bound for a sparse subclass of $\mathfrak{M}$ and for two- and three-component systems. Also C. Newman kindly pointed out to me that he proved the magnetization bound for one-component systems for which GHS inequalities hold—a result announced at Rutgers University in December 1981.

In the one-component case the proof (Section 2) is based on an extension of Jensen's inequality (Section 3), to odd functions which are

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concave on the positive half-axis and to measures μ such that μ ($[a + \infty[$) $\geqslant \mu(]-\infty, -a])$ for any positive a. That the measures defined by ferromagnetic systems satisfy this condition is shown in Section 4 using the FKG inequalities. Many-component systems are treated in Section 5, with a proof of the needed convexity properties of Bessel functions sketched in Appendix B. We consider systems on lattices more general than $\mathbb{Z}^\nu$, with a few points per unit cell, like those on face centered lattices. This leads to a system of mean field equations properties of which may be known but for which we found no references. This is discussed in Appendix A. The interactions are not assumed to be of a finite range.

2. ONE-COMPONENT MODELS

Our lattice $\mathbb{L}$ is a discrete $\mathbb{Z}^\nu$-invarient subset of $\mathbb{R}^\nu$. The spin distributions μ_a, the external field h_a, $a \in \mathbb{L}$, and the two-body interaction $(J(a, b))_{a,b\in\mathbb{L}}$ are $\mathbb{Z}^\nu$-invariant. Furthermore the measures μ_a on $\mathbb{R}$ are assumed to be even of a compact support and not concentrated at $\{0\}$, $J(a,b), h_a \geqslant 0$,

$$\sum_b J(a,b) < \infty, \qquad \text{any } a$$

The lattice is assumed to be J-connected, i.e., for any $a, b \in \mathbb{L}$ there is a natural n and $a_1, \ldots, a_n \in \mathbb{L}$, $a_1 = a$, $a_n = b$, such that $J(a_i, a_{i+1}) \neq 0$, $i = 1, \ldots, n-1$. Any model can be reduced to a family of connected models by passing to "J-components" of $\mathbb{L}$. The configuration space of the system is

$$\mathfrak{X} = \prod_{a\in\mathbb{L}} [-r_a, r_a], \qquad \mathfrak{X}_\Lambda = \prod_{a\in\Lambda} [-r_a, r_a]$$

where $r_a = \sup\operatorname{supp} \mu_a : s_a \cdot \mathfrak{X} \to \mathbb{R}$ are the usual spin variables, i.e., s_a is the projection on the ath coordinate. The (ferromagnetic) Hamiltonian H is written as

$$H = -\sum_{a,b} J(a,b) s_a s_b - \sum_a h(a) s_a$$

For an inverse temperature β we set, as usual,

$$K(a,b) = \beta J(a,b), \qquad k(a) = \beta h(a)$$

$$p(K,k) = \lim \frac{1}{|\Lambda|} \log Z_\Lambda(K,k)$$

$$Z_\Lambda = \int\int \exp\left[\sum_{a,b\in\Lambda} K(a,b) s_a s_b + \sum_a k(a) s_a\right] \bigotimes_{a\in\Lambda} d\mu_a$$

The magnetization $(m_a)_{a\in\mathbb{L}}$ is the right derivative of p:

$$m_a = \lim_{\epsilon\downarrow 0} \frac{p(\ldots,k_a+\epsilon,\ldots)-p(\ldots,k_a,\ldots)}{\epsilon}$$

By the well known properties of equilibrium states

$$m_a = \langle s_a\rangle^+$$

where $\langle\ \rangle^+$ is the equilibrium state obtained as the limit of the finite volume states $\langle\ \rangle^+$ defined by the "+" boundary conditions. The $\langle\ \rangle^+$ state inherits the symmetries of the Hamiltonian. In particular, it is $\mathbb{Z}^\nu$ invariant.

Let for $a \in \mathbb{L}$

$$f_a(x) = \frac{\int te^{tx}\mu_a(dt)}{\int e^{tx}\mu_a(dt)}$$

Then the DLR conditions are that, in particular,

$$\langle s_a\rangle^+ = \left\langle f_a\left(\sum K(a,b)s_b + k(a)\right)\right\rangle^+, \qquad \text{all} \quad a\in\mathbb{L}$$

The functions f_a are odd. They are strictly increasing since

$$f_a'(x) = \frac{\int (t-f_a(x))^2\mu_a(dt)}{\int e^{tx}\mu_a(dt)} > 0$$

Let us *assume that* f_a *are of class* $\mathfrak{M}$,[1,4] i.e., that they are concave on $[0,+\infty]$ (cf. Fig. 1); the class $\mathfrak{M}$ is discussed at the end of this section. Then by Sections 3 and 4 the following *fundamental inequality* holds:

$$\langle s_a\rangle^+ \leqslant f_a\left(\left\langle \sum_a K(ab)s_b + k(a)\right\rangle^+\right) \tag{1}$$

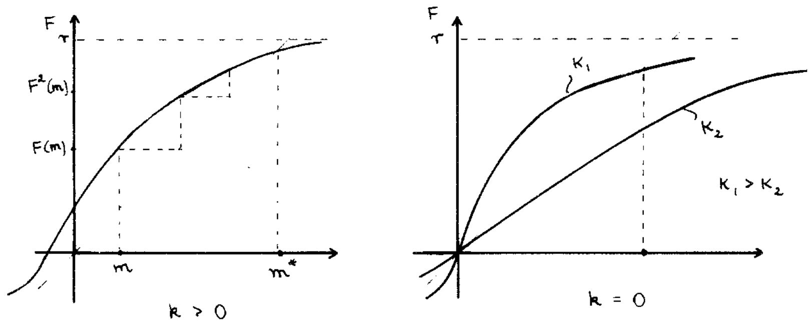

Fig. 1. Fixed points of F for $k>0$ and for $k=0$. For $k>0$ the iterates $F(m)$, $F^2(m)$ $F^3(m)$ are indicated.

Thus, setting

$$m_a = \langle s_a \rangle^+$$

we have $m_a \geqslant 0$, (m_a) is $\mathbb{Z}^\nu$ invariant, and

$$m_a \leqslant f_a\Big(\sum_b K(a,b)m_b + k(a)\Big) \qquad (\text{all } a)$$

The mean field bound for magnetization is an easy consequence of this. We first discuss the case of $\mathbb{L} = \mathbb{Z}^\nu$ when the arguments have a simple geometric interpretation (Fig. 1). Then we summarize the generalization to general lattices worked out in Appendix A.

If $\mathbb{L} = \mathbb{Z}^\nu$, m_a, f_a, $k(a)$ and $K(a) = \sum_b K(a,b)$ are a independent and, dropping the a, (1) becomes

$$m \leqslant f(Km + k)$$

It is obvious from Fig. 1 that for $k > 0$ there is only one nonnegative solution m^* of the equation

$$m = F(m), \qquad m \geqslant 0$$

where $F(m) = f(Km + k)$. It is given by the intersection of graphs of F and $m \mapsto m$.

The fixed point of F has the following properties.

For $k = 0$ there is always the zero fixed point. For small K it is the only fixed point. As K increases this fixed point becomes unstable and eventually there appears another strictly positive stable solution. The critical, or bifurcation, value of K is given by $F'(0) = 1$, i.e., $K_{\rm cr} = f'(0)^{-1}$. Thus for any (K, k) there is only one nonnegative stable fixed point m^* of F. It depends on (K, k) in a continuous and monotonic way. If $m \leqslant F(m)$ then $m \leqslant m^*$.

In Appendix A we analyze the general case, with conclusions similar to the above. In the space M^I of $\mathbb{Z}^\nu$-invariant nonnegative m's, $m = (m_a)_{a \in \mathbb{L}}$, we consider the map $Fm \to (f_a(\sum_b K(a,b)m_b + k_a))a \in \mathbb{L}$. For $k = 0$ and $\beta \leqslant \beta_{\rm cr}$ there is the zero fixed point only. It becomes unstable at $\beta_{\rm cr}$ and for $\beta > \beta_{\rm cr}$ there is a unique nonzero fixed point. Thus for each (K, k) there is unique maximal fixed point m^* of F. It depends on (K, k) in a continuous and monotonic way, and *if* $m_a \leqslant F(m)_a$ *then* $m_a \leqslant m_a^*$ *and, for* $m0$, $F^n(m) \to m^*$ *as* $n \to \infty$. $\beta_{\rm cr}$ is given by the condition that the maximal eigenvalue of $F'(0)$ is equal to 1, i.e., $\beta_{\rm cr}$ is the inverse of the maximal eigenvalue of the map

$$m \to \Big(f_a'(0) \sum_b J(a,b) m_b\Big)_{a \in \mathbb{L}}$$

(m is $\mathbb{Z}^\nu$ invariant). Thus (1) implies the mean field bound

$$m_a \leqslant m_a^*, \qquad a \in \mathbb{L}$$

This completes the discussion in the case $\mu_a \in \mathfrak{M}$.

We now discuss the class $\mathfrak{M}$ to which our results apply. For spin p, $p = 1/2, 1, 3/2, \ldots$

$$\mu = \sum_{r=-p,\,-p+1,\,\ldots,\,p} \delta_r$$

$$f(x) = \frac{2p+1}{2}\operatorname{cth}\frac{2p+1}{2}x - \frac{1}{2}\operatorname{cth}\frac{x}{2} = pB_p(px)$$

where B_p is the corresponding Brillouin function. By direct calculation one checks that $f''(x)$ is negative for $x > 0$. Thus our results apply to one-component spin systems with arbitrary value of the spin. Pearce[4] proves the mean field bound for $2p + 1 = 2^q \cdot 3^r$ with natural q, r.

In case of a continuous distribution: $\mu(dx) = \rho(x)dx$, ρ supported by $[-1, 1]$ and not decreasing on $[0, 1]$, Pearce proves a property stronger than $\mathfrak{M}$. This allows him to treat D-component systems (cf. Section 5) with $D = 2, 3$. In Appendix B we sketch a proof that the measure μ_D,

$$\int f(x)\mu_D(dx) = \int_{-1}^{1} f(x)(1 - x^2)^{(1/2)(D-3)}dx$$

is of class $\mathfrak{M}$ for any D. This yields the mean field bound for any number of components (Section 5).

3. A VERSION OF THE JENSEN'S INEQUALITY

The following proposition has been used in the preceding section.

Proposition. Let ρ be a probability measure on a space $\mathfrak{X}$ and let X be a real random variable. Let f be an odd function $\mathbb{R} \to \mathbb{R}$ which is concave on $[0, +\infty[$. If for any $a \geqslant 0$ $\rho(X \geqslant a) \geqslant \rho(X \leqslant -a)$ and both X and $\phi \circ X$ are ρ-integrable then

$$\int f \circ X\,d\rho \leqslant f\Big(\int X\,d\rho\Big)$$

Using a standard approximation argument we first reduce our problem to one with X assuming only a finite number of values: Let

$$\begin{aligned} X_n(x) &= \pm\frac{k}{2^n} \qquad \text{if} \quad \pm X(x) \in \Big[\frac{k}{2^n}, \frac{k+1}{2^n}\Big[, \qquad k = 0, \ldots, 2^{2n} \\ &\ \pm 2^n \qquad \text{if} \quad \pm X(x) \geqslant 2^n \end{aligned}$$

Clearly $\rho(X_n \geqslant a) \geqslant \rho(X_n \leqslant -a)$, any $a \geqslant 0$, and therefore by the argument of Ref. 2, p. 303, it is enough to prove the proposition for each X_n.

Thus we can assume that there are nonnegative numbers $x_0 < \cdots < x_m$, $x_0 = 0$; $y_1 < y_n$, such that, with $\alpha_i = \rho(X = x_i)$, $\beta_i = \rho(X = -y_i)$, $\sum_i \alpha_i + \sum_i \beta_i = 1$. Then under our assumptions on f we have to prove that

$$f\Big(\sum_i \alpha_i x_i - \sum_j \beta_j y_j\Big) \geqslant \sum_i \alpha_i f(x_i) - \sum_j \beta_j f(y_j)$$

if for any $a \geqslant 0$

$$\sum_{x_i \geqslant a} \alpha_i \geqslant \sum_{y_j \geqslant a} \beta_j$$

Since the term x_0 contributes zero to both sides of the inequality we will take $i \geqslant 1$ and assume

$$\sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{n} \beta_i \leqslant 1$$

If all $\beta_i = 0$ this is just an expression of the concavity. In case $m, n = 1$ the inequality has the graphical interpretation shown in Fig. 2. The interval joining $(x, f(x))$ with $(-y, f(-y))$ lies below the interval joining $(x, f(x))$ with $(-x, f(-x))$, which in turn lies below the graph of f for the values of the argument between 0 and x. The following arguments implement the above idea.

I will show below that under our conditions one can decompose β_j as follows: there are $(\gamma_{ij})_{i=1,\ldots,m; j=1,\ldots,n}$ such that $\gamma_{ij} \geqslant 0$, $\sum_i \gamma_{ij} = \beta_j$, $\sum_j \gamma_{ij} = \alpha_i$ and $\gamma_{ij} = 0$ if $x_i < y_j$. Assuming this rather intuitive fact the proof

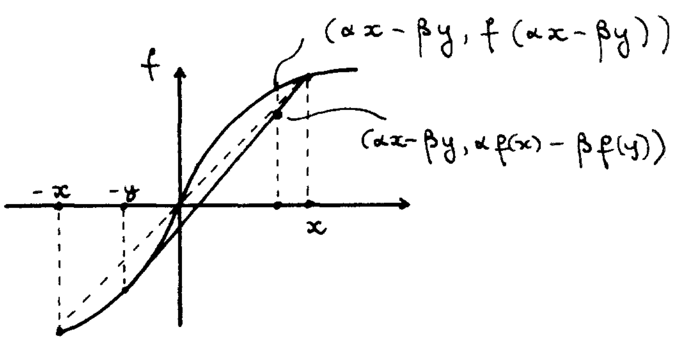

Fig. 2. To the proof of a version of the Jensen's inequality.

continues as follows:

$$\begin{aligned}
f\Big(\sum \alpha_i x_i - \sum_j \beta_j y_j\Big) &= f\Big(\sum_i \alpha_i x_i - \sum_j y_j \sum_i \gamma_{ij}\Big) \\
&= f\Big(\sum_i \alpha_i x_i - \sum_i x_i \sum_j \frac{y_j}{x_i}\gamma_{ij}\Big) \\
&= f\Big(\sum_i \Big(\alpha_i - \sum_j \frac{y_j}{x_i}\gamma_{ij}\Big)x_i\Big) \geqslant \sum_i \Big(\alpha_i - \sum_j \frac{y_j}{x_i}\gamma_{ij}\Big) f(x_i) \\
&= \sum_i \alpha_i f(x_i) - \sum_j \sum_i \gamma_{ji}\frac{y_j}{x_i} f(x_i) \\
&\geqslant \sum_i \alpha_i f(x_i) - \sum_j \sum_i \gamma_{ij} f(y_j) = \sum_i \alpha_i f(x_i) - \sum_j \beta_j f(y_j)
\end{aligned}$$

Here to obtain the first inequality use the fact that γ_{ij} is zero if $y_j/x_i > 1$ and that therefore $\alpha_i - \sum_j \gamma_{ij} y_j/x_i \geqslant 0$ and $\sum_i[\alpha_i - \sum_j(y_j/x_i)\gamma_{ij}] \leqslant 1$.

It remains to demonstrate the existence of (γ_{ij}). This is done by induction with respect to n as follows.

Let $k < m$ be the largest integer with $\sum_{i>k}\alpha_i \geqslant \beta_n$; that such k exists and that $x_k \geqslant y_n$ follows from the condition (3). Define now

$$\gamma_{i,n} = \begin{cases} \alpha_i & \text{if } \; i > k \\ 0 & \text{if } \; i < k \end{cases}$$

$$\gamma_{k,n} = \beta_n - \sum_{i>k} \alpha_i$$

(here α_i is set equal to 0 for $i > m$). Obviously, $\gamma_{i,n}$ has the correct properties.

Define now $\alpha_i^{(1)} = \alpha_i - \gamma_{i,n}$. Then the system $(x_1, \alpha_1^{(1)}; x_2, \alpha_2^{(1)}; \dots ; y_1, \beta_1; \dots ; y_{n-1}, \beta_{n-1})$ again satisfies the condition (3) and we can continue with construction of $\gamma_{i,n-1}$. Existence of γ_{ij} is proved.

Joel Lebowitz pointed out to me that (γ_{ij}) looks very much like the measure ν (Ref. 5, p. 185) of a proof of the FKG inequalities.

4. THE FUNDAMENTAL INEQUALITY

Assuming now that f_a of Section 2 is of class $\mathfrak{M}$ we will use the FKG inequalities (Ref. 5, Chap. 3) to deduce the fundamental inequality from Proposition 3.

We note that if X_n is a sequence of random variables satisfying (2) and $X_n \to X$ pointwise then X satisfies (2). Also if ρ_n satisfies (2), $\rho_n(X \geqslant a) \to \rho(X \geqslant a)$ and $\rho_n(X \leqslant -a) \to \rho(X \leqslant -a)$, $a \geqslant 0$, then ρ satisfies (2).

Therefore it is enough to show that with $\mathfrak{X} = \mathfrak{X}_\Lambda$, $\rho = \langle\ \rangle_\Lambda^+$, $X = \sum_{b \in \Lambda} K(a,b) s_b$, $a \in \Lambda$, the condition (2) is satisfied.

Let ρ^0 be the Gibbs state on $\mathfrak{X}_\Lambda$ corresponding to $k = 0$ and zero boundary conditions. Then by the up–down symmetry

$$\rho^0(X \geqslant a) = \rho^0(X \leqslant -a)$$

Since ρ is obtained from ρ^0 by increasing the interaction in the sense of FKG and for positive $K(a,b)$ the characteristic function of $X \geqslant a$ (resp. $X \leqslant -a$) is FKG increasing (resp. FKG decreasing) one obtains

$$\rho(X \geqslant a) \geqslant \rho^0(X \geqslant a) = \rho^0(X \leqslant -a) \geqslant \rho(X \leqslant -a)$$

as required. The rest follows from the fact that under our assumptions for each $a \in \mathbb{L}$ the sum

$$X = \sum_{b \in \mathbb{L}} K(a,b) s_b + k(a)$$

is convergent.

5. MANY COMPONENTS

We first explain the form of the mean field bound.

Consider the $\mathbb{L} = \mathbb{Z}^\nu$ case; generalization to the general case is obvious. Then the mean field magnetization solves the equation

$$\bar{m} = \frac{\int \mathbf{s} e^{\bar{s}(K\bar{m} + \bar{k})} \, d\Omega(\bar{s})}{\int e^{\bar{s}(K\bar{m} + \bar{k})} \, d\Omega(\bar{s})}$$

where $K = \sum_b \beta J(a,b)$, $\bar{k} = \beta \bar{h}$, $\bar{h}$ is the external field and $d\Omega$ is a rotation-invariant measure on the unit sphere of $\mathbb{R}^D$. With

$$\Phi(\bar{y}) = \int e^{\bar{s} \cdot \bar{y}} \, d\Omega(s)$$

we have

$$\bar{m} = \frac{\nabla \Phi}{\Phi} (K\bar{m} + \bar{k})$$

Now, because of rotation invariance, $\Phi(\bar{y}) = \phi(|\bar{y}|)$ where ϕ is a function of one variable. Thus

$$\bar{m} = \frac{\phi'}{\phi} \left(|K\bar{m} + \bar{k}| \right) \frac{K\bar{m} + \bar{k}}{|K\bar{m} + \bar{k}|}$$

It follows that, for $\bar{k} \neq 0$, $\bar{m}$ is parallel to $\bar{k}$ and if it has direction of $\bar{k}$, as in the case of interest here, then

$$m = \frac{\phi'}{\phi}(Km + k), \qquad m = |\overline{m}| \tag{4}$$

This has the form of the mean field equation for one-component models and the proof that the solution of (4) bounds the magnetization of the many-component model is through first comparing it[(4)] with the corresponding one-component model and then using mean field majorization for the later.

Let

$$p(k) = \lim_{\Lambda} p_\Lambda(k), \qquad p_\Lambda(k) = \frac{1}{|\Lambda|} \log Z_\Lambda(k)$$

$$Z_\Lambda = \int \int \exp\left[\sum_{a,b \in \Lambda} K(a,b) \mathbf{s}_a \cdot \mathbf{s}_b + k \sum_{a \in \Lambda} s_a^1 \right] \bigotimes_{a \in \Lambda} d\Omega(\mathbf{s}_a)$$

Then, by definition

$$m = \lim_{\epsilon \downarrow 0} \frac{p(k+\epsilon) - p(k)}{\epsilon} = D^+ p(k)$$

Let $\tilde{k} \geqslant k$ and let D stand for the derivative; by the convexity of p and p_Λ

$$m(k) \leqslant \lim D p_{\Lambda_n}(\tilde{k})$$

where the (expanding) sequence (Λ_n) has been so chosen that $Dp_{\Lambda_n}(\tilde{k})$ is convergent.

On the other hand by Ref. 4, Section 3, $Dp_{\Lambda_n}(\tilde{k}) \leqslant D\tilde{p}_{\Lambda_n}(\tilde{k})$ where $\tilde{p}(\tilde{k})$ corresponds to a one-component system with

$$-\beta H = \sum_{ab} K(a,b) s_a s_b + k \sum_a s_a$$

and μ_a such that

$$\int f(s) \mu_a(ds) = \int f(s^1)\, d\Omega(\mathbf{s}) \tag{5}$$

Passing to a subsequence $(\Lambda_{n_k})_{k=1}^{\infty}$ for which $D\tilde{p}_\Lambda(\tilde{k})$ is convergent we see that there exists an equilibrium state of the one-component system, say, ρ, such

$$Dp_{\Lambda_{n_k}}(\tilde{k}) \to \rho(s_a)$$

and thus

$$m(k) \leqslant \rho(s_a)$$

By the maximality property of the "+" state, $\rho(s_a) \leqslant \rho^+(s_a)$ and since

$\rho_{\tilde{k}}^{+}(s_a) \geqslant \rho_k^{+}(s_a)$ as $\tilde{k} \geqslant k$ we obtain the majorization

$$m(k) \leqslant \rho_k^{+}(s_a)$$

According to Appendix B the measure μ_a defined by (5) is of class $\mathfrak{M}$. Therefore, by Section 2, $\rho_k^{+}(s_a)$ is majorized by the maximal nonnegative solution $m(K,k)$ of (4) which is the mean field bound.

All this generalizes to more general lattices, as in Section 2, with the equation (4) replaced by

$$m_a = \frac{\phi'}{\phi}\left(\sum_b K(a,b)m_b + k_a\right), \qquad a \in \mathbb{L}$$

ACKNOWLEDGMENT

I would like to thank Joel Lebowitz for helpful and stimulating discussions, and for pointing out to an error in an earlier version of this work. Discussions with my colleagues at VPI on the subject of Appendix B are also gratefully acknowledged.

APPENDIX A. THE MF MAGNETIZATION

The framework is as in Section 2. M is the set of all $m = (m_a)_{a\in\mathbb{L}}$, $0 \geqslant m_a \geqslant 1$. $m \leqslant n$, $m, n \in M$, if $m_a \leqslant n_a$ $\forall a \in \mathbb{L}$, and $m < n$ if $m \leqslant n$ and $m \neq n$. A sequence $m^{(n)}$ in M is converging to m if $m_a^{(n)} \to m_a$, $\forall a \in \mathbb{L}$.

$F: M \to M$ is defined by

$$F(m)_a = f_a\left(\beta \sum_b J(a,b)m_b + \beta h_a m_a\right)$$

We will use the following monotonicity and convexity properties of F:

- (F_1) F is strictly increasing, i.e., $m < n \Rightarrow F(m) < F(n)$.

This follows from positivity of J and h and from the fact that f_a's are strictly increasing.

Let M^I be the set of all $\mathbb{Z}^\nu$-invariant elements of M. M^I is F invariant and of a finite dimension equal to $|\mathbb{L}/\mathbb{Z}^\nu|$. The set M_*^I of elements of M^I which are nowhere zero is invariant too.

- (F_2) For any $m \in M_*^I$ and $0 < t < 1$ there is $\eta > 0$ such that

$$F(tm) \geqslant (1+\eta)tF(u)$$

For $\eta = 0$ this is a direct consequence of concavity of f_a's. With $\eta > 0$ this is an easy consequence of the strict convexity of f_a's (cf. below). In case $k_a > 0$, $\forall a$, (F_2) holds for arbitrary $m \in M$, by convexity and strict monotonicity of f_a's.

To demonstrate the strict convexity of f_a it is enough to show that $f_a''(x)$ which is non-negative for $x > 0$, vanishes at isolated points only. But f_a extends to an analytic function in a complex neighborhood of the real axis. Therefore, the same holds for f_a''. If the real zeros of f_a'' were not isolated it would vanish identically and f_a itself would be a linear function, which it is not.

We consider first the solutions of

$$m = F(m), \qquad m \in M \tag{A1}$$

The property (F_2) is not used in the lemma below.

Lemma (and definition):

1. If $m \leqslant F(m)$ then the sequence $F^k(m)$ is convergent. Its limit m^* is a solution of (A1). Furthermore $m \leqslant m^*$.
2. If m is a nonzero solution of (A1) then $m_a \neq 0\ \forall a \in \mathbb{L}$.
3. For any family (m_i) of solutions of (A1) there is a solution majorizing all of m_i's.
4. There exists a maximal solution $m(K)$ of (A1); here $K = (\beta J(a,b), \beta h(a))_{a,b \in \mathbb{L}}$. If $K > K'$ then $m(K) > m(K')$. If $K^{(n)} \downarrow K$ then $m(K^{(n)}) \downarrow m(K)$.
5. The maximal solution is $\mathbb{Z}^\nu$ invariant.

Proof. By (F1), $F^{k+1}(m) > F^k(m)$. Since $F^k(m)_a = \leqslant 1$, No. 1 follows. No. 2 holds by the J connectedness of $\mathbb{L}$. With $m = \sup m_i$, $m^* > m > m_i$, all i, which proves No. 3.

Let $m(K)$ be the supremum of all solutions of (A1). By No. 3, $m(K)$ is the maximal solution. If $K \geqslant K'$ then $m(K') = F_{K'}(m(K')) \leqslant F_K(m(K'))$ and therefore $m(K) \geqslant m(K')$ by No. 1.

Let $m = \lim m(K^{(n)})$. Then since $m(K^{(n)}) \geqslant m(K)$, $m \geqslant m(K)$. But by continuity of K, $m \mapsto F_K(m)$, $m = F_K(m)$. Thus $m = m(K)$, and No. 4 is proved. No. 5 follows from the fact that a translate of a solution is again a solution and from uniqueness of the maximal solution.

Proposition. $m(K) \neq 0$ for $h \neq 0$. If $h = 0$, $m(K) \neq 0$ for $\beta > \beta_{\text{cr}}$ and $m(K) = 0$ for $\beta \leqslant \beta_{\text{cr}}$. β_{cr} is given by the condition that the maximal eigenvalue of the derivative of $F \restriction M^I$ at zero is 1.

Assume now $h = 0$ and consider the restriction of F to M^I, denoted again by F. Let F' be its derivative at 0:

$$(F'm)_a = f_a'(0) \qquad \sum_b \beta J(a,b) m_b$$

Thus in the natural basis all matrix elements of F' are nonnegative. Moreover, since $\mathbb{L}$ is J-connected there is a power of F which has all matrix elements strictly positive. By the Perron–Frobenius theorem the eigenvalue

λ_{max} of F which is maximal in modulus is positive nondegenerate and the corresponding eigenvector $\tilde{m}$ can be chosen to be (strictly) positive. It is obvious from the form of F that $\tilde{m}$ does not depend on β and that λ_{max} is proportional to β:

$$\lambda_{max}(\beta) = \tilde{\lambda} \cdot \beta$$

Assume $\lambda_{max} > 1$. Then $F'\tilde{m} = \lambda_{max}\tilde{m} \geqslant \tilde{m}$ in the sense that $(F'm)_a \geqslant m_a$, any $a \in \mathbb{L}$. Since $t^{-1}F(t\tilde{m}) \to F'\tilde{m}$ as $t \to 0$ it follows that for small enough $t, t^{-1}F(t_2\tilde{m}) \geqslant \tilde{m}$. Thus for $\lambda_{max} > 1$, $(t\tilde{m})^*$ is a nonzero solution of (A1).

Next we note that for any $m \in M^I$, $m \neq 0$, $1 \geqslant t_1 > t_2 > 0$

$$F(m) \leqslant t_1^{-1}F(t_1 m) < t_2^{-1}F(t_2 m)$$

the strict inequality following from (F_2). It follows that

$$F(m) < F'm \tag{A2}$$

Now if $m \in M^I$ is a nonzero fixed point of F and $\tilde{n}$ is the positive eigenvector of the matrix transposed to F' corresponding to the eigenvalue λ_{max} then, taking (A2) into account,

$$(\tilde{n}, m) = (\tilde{n}, F(m)) < (\tilde{n}, F'm) = \lambda_{max}(\tilde{n}, m)$$

which is impossible if $\lambda_{max} \leqslant 1$. Thus it has been shown that $m(K)$ is nonzero if $\tilde{\lambda} . \beta > 1$ and zero if $\tilde{\lambda} . \beta \leqslant 1$. This yields $\beta_{cr} = \tilde{\lambda}^{-1}$.

In fact in our situation (A1) has unique solution if $h \neq 0$ and at most two $\mathbb{Z}^\nu$-invariant solutions if $h = 0$: the zero solution, and unique nonzero solution, if it exists.

The proof below and the form of the condition (F_2) are adapted from Ref. 3. Let m be a fixed point of F. In both cases to be considered, there is a constant $\alpha > 0$ depending on m, such that $m_a \geqslant \alpha$, $\forall a \in \mathbb{L}$; this follows from J-convectivity of $\mathbb{L}$. Let n be another fixed point with corresponding constant $\beta > 0$. Suppose it is not true that $m \geqslant n$, and let $t_0 = \sup\{t \geqslant 0 : m \geqslant tn\}$. Since $n_a \leqslant 1$ and $m \not\geqslant n$, $0 < \alpha \leqslant t_0 < 1$. But then, using monotonicity of F and (F_2),

$$m = F(m) \geqslant F(t_0 n) \geqslant (1 + \eta)t_0 F(n) = (1 + \eta)t_0 n$$

Thus $m \geqslant (1 + \eta)t_0 n$ is contradiction with the maximality of t_0.

APPENDIX B. CONVEXITY FOR MANY COMPONENTS

With suitable normalization of the rotation-invariant measure $d\Omega$ on the unit sphere of $\mathbb{R}^D$ the measure μ_a of (15) satisfies

$$\int f(s)\mu_a(ds) = \int_{-1}^{1} (1 - s^2)^{(1/2)(D-3)} f(s)\, ds$$

and thus the ϕ of (4) is

$$\phi(x) = \int_1^1 e^{x \cdot s}(1 - s^2)^{(1/2)(D-3)}\, ds:$$

We have to show that $f = \phi'/\phi$ is convex on the positive half-axis.

f is obviously an analytic function. Also

$$f(x) = J_{D/2}(x)/J_{(D/2)-1}(x) \tag{B1}$$

where J is the modified Bessel function.[4,6] Either by direct integration by parts or from the well-known identities

$$J_n' = J_{n+1} + \frac{n}{x} J_n, \qquad J_n' = J_{n-1} - \frac{n}{x} J_n$$

one can see that f satisfies the differential equation

$$f' = 1 - \frac{D-1}{x} f - f^2 \qquad \text{with} \quad f(0) = 0 \tag{B2}$$

From (B1) or (B2) one obtains easily an expansion of f for small x.

$$f(x) = \frac{x}{D}\left[1 - \frac{x^2}{D(D+2)}\right] + O(x^5) \tag{B3}$$

We first show that $0 < f(x) < 1$, $0 < f' < -1$ as $0 < x < +\infty$. Let

$$F(x, f) = 1 - \frac{D-1}{x} f - f^2, \qquad x, f \geqslant 0$$

Then as is easy to see the situation is as in Fig. 3. The curve $F(x, f) = 0$ is concave and asymptotic to $f = 1$ as $x \to \infty$. The slope of f at 0 is $1/D$ [from (B3)], whereas the slope of $F(x, f) = 0$ at 0 is $1/(D-1)$. Thus for small x, the graph of f is between the x axis and the curve $F(x, f) = 0$. But then it remains in this region for all $x > 0$, i.e., $f'(x) > 0\ \forall x > 0$. For otherwise the graph of f would have to cross the curve $F(x, f) = 0$. This is impossible as the slope of the last curve is positive everywhere, whereas f at the crossing point would have slope 0.

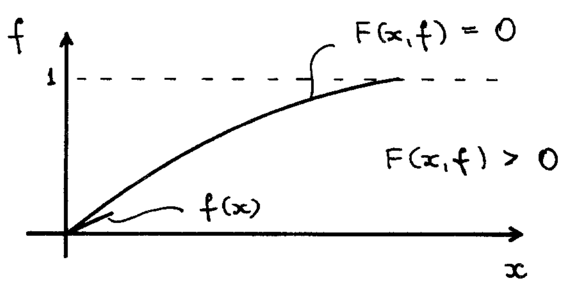

Fig. 3. Behavior of the solution of the equation (B.2).

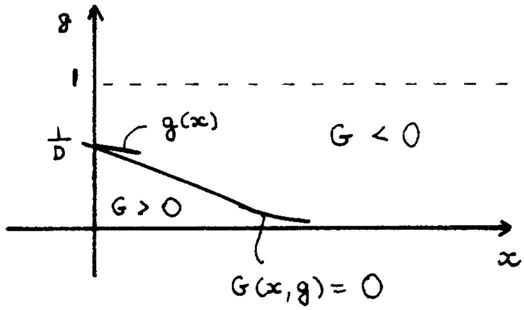

Fig. 4. Behavior of $g = f'$.

We use a similar argument to show that f'' is negative for $x > 0$. Differentiating (B2) and eliminating f we see that

$$f''(x) = \frac{D-1}{2x^2}\left\{-\frac{D-1}{x} + \left[\left(\frac{D-1}{x}\right)^2 - 4(f'-1)\right]^{1/2}\right\}$$
$$-f'\left[\left(\frac{D-1}{x}\right)^2 - 4(f'-1)\right]^{1/2}$$

Or, with $g = f'$, $d = D - 1$

$$g' = G(x, g)$$

where

$$G(x, g) = \frac{d}{2x^2}\left\{-\frac{d}{x} + \left[\frac{d^2}{x^2} - 4(g-1)\right]^{1/2}\right\} - g\left[\frac{d^2}{x^2} - 4(g-1)\right]^{1/2}$$

Somewhat more involved analysis than before yields now the graph of Fig. 4. The curve $G(x, g) = 0$ has negative slope. For small positive x the graph of g is above this curve, i.e., in the region $G < 0$. But then it stays above for all $x > 0$ by the argument used in analyzing the graph of f. Hence $f'' = g' = G(x, g) < 0$ for all x. Which demonstrates the (strict) concavity of f.

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